1.5: Linear First-Order Equations

Definition 1. A **linear first-order** differential equation is of the form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1}$$

In order to solve an equation of the form in (1), we must use the **integrating factor**

$$\rho(x) = e^{\int P(x)dx}. \tag{2}$$

<u>Solution of Linear First-Order Equations</u>

1. Begin by calculating the integrating factor $\rho(x) = e^{\int P(x)dx}$ in (2).
2. Multiply both sides of the equation by $\rho(x)$.
3. Integrate both sides of the equation.
   (a) Recognize that the left-hand side of the equation is the derivative $D_x(\rho(x)y(x))$.
   (b) Integrate the right-hand side of the equation as usual.

Example 1. Solve the initial value problem

$$\frac{dy}{dx} - y = \tfrac{11}{8}e^{-x/3}, \quad y(0) = -1.$$

Example 2. Find a general solution of

$$(x^2+1)\frac{dy}{dx}+3xy=6x.$$

Theorem 1. (Linear First-Order Equation) If the functions $P(x)$ and $Q(x)$ in (1) are continuous on the open interval I containing the point x_0, then the initial value problem

$$\frac{dy}{dx}+P(x)y=Q(x), \quad y(x_0)=y_0$$

has a unique solution $y(x)$ on I.

Exercise 1. Solve the initial value problem

$$x^2\frac{dy}{dx}=\sin x-xy, \quad y(1)=y_0.$$

Homework. 1-25 (odd)